MA 3046 - Matrix Analysis

Problem Set 5 - Conditioning and Stability (Partial Set)

1. Consider each of the following systems from each of three perspectives -
   (i) As a computational problem to be solved by Gaussian Elimination in a three digit, rounding machine (you may wish to check your answers with the laboratory routine **ge_steps_chop.m**)
   (ii) As a geometric problem involving the intersection of two lines
   (iii) As a geometric problem involving construction of one (column) vector in terms of two others.

(a.) $$\begin{array}{rcrcl} 2.01\,x_1 & - & 1.99\,x_2 & = & 4.00 \\ 1.99\,x_1 & + & 2.01\,x_2 & = & 4.00 \end{array}$$

(b.) $$\begin{array}{rcrcl} 2.01\,x_1 & + & 1.99\,x_2 & = & 4.00 \\ 1.99\,x_1 & + & 2.01\,x_2 & = & 4.00 \end{array}$$

(c.) $$\begin{array}{rcrcl} 1.01\,x_1 & + & 0.01\,x_2 & = & 4.00 \\ 0.01\,x_1 & + & 1.01\,x_2 & = & 2.00 \end{array}$$

Which of these would you expect to be ill-conditioned and which to be well conditioned? What attributes appear to be associated with well-conditioned problems from each geometric view?

2. Using MATLAB to calculate the inverses in each case, determine the condition number (using the infinity norm) for each of the matrices in problems 1. Do these values support your geometric intuition about which problems are and are not well-conditioned?

3. Based on your results from problems 1 and 2, what geometric attributes would you associate with the best-conditioned systems of two linear equations in two unknowns?

4. Use MATLAB to generate random 6×6 matrices with condition numbers satisfying:
   a. $1 < \text{cond}(\mathbf{A}) \leq 10$
   b. $10 < \text{cond}(\mathbf{A}) \leq 100$
   c. $100 < \text{cond}(\mathbf{A}) \leq 1000$
   d. $1000 < \text{cond}(\mathbf{A})$

For each of the matrices, have MATLAB compute $\mathbf{A}^{-1}$, and observe the correlations, if any, between $\text{cond}(\mathbf{A})$, the elements of $\mathbf{A}$, and the elements of $\mathbf{A}^{-1}$.

5. Consider the two systems

$$\begin{array}{rcrcrcl} 4.55\ x_1 & + & 2.39\ x_2 & + & 4.18\ x_3 & = & 7.52 \\ 2.40\ x_1 & + & 2.04\ x_2 & + & 3.75\ x_3 & = & 4.77 \\ 3.19\ x_1 & - & 2.06\ x_2 & - & 4.23\ x_3 & = & 1.45 \end{array}$$

and

$$\begin{array}{rcrcrcl} 4.55\ x_1 & + & 2.39\ x_2 & + & 4.18\ x_3 & = & 7.56 \\ 2.40\ x_1 & + & 2.04\ x_2 & + & 3.75\ x_3 & = & 4.72 \\ 3.19\ x_1 & - & 2.06\ x_2 & - & 4.23\ x_3 & = & 1.50 \end{array}$$

(Notice the left-hand sides here are identical, and the right-hand side of the second is only a "relatively small" perturbation of the right-hand side of the first.)

a. Compare the exact (MATLAB) solutions to both problems. What does that comparison suggest about the condition of this matrix?

b. Using MATLAB, calculate the condition number of the matrix in this problem. Does your result support your conclusion in part a or not?

6. Consider the system of equations:

$$\begin{array}{rcrcrcrcl} x_1 & + & \frac{1}{2}\ x_2 & + & \frac{1}{3}\ x_3 & + & \frac{1}{4}\ x_4 & = & \frac{4}{3} \\[6pt] \frac{1}{2}\ x_1 & + & \frac{1}{3}\ x_2 & + & \frac{1}{4}\ x_3 & + & \frac{1}{5}\ x_4 & = & \frac{5}{6} \\[6pt] \frac{1}{3}\ x_1 & + & \frac{1}{4}\ x_2 & + & \frac{1}{5}\ x_3 & + & \frac{1}{6}\ x_4 & = & \frac{19}{30} \\[6pt] \frac{1}{4}\ x_1 & + & \frac{1}{5}\ x_2 & + & \frac{1}{6}\ x_3 & + & \frac{1}{7}\ x_4 & = & \frac{31}{60} \end{array}$$

(The matrix here is an example of the so-called *Hilbert Matrix*, and is classic in numerical analysis.)

(a.) Simulate the solution of this system using Gaussian elimination on a three-digit, decimal based computer which rounds all calculations, including intermediate ones. (Note the true solution here is $\mathbf{x} = [1 \quad -2 \quad 4 \quad 0]^T$.

(b.) For your computed solution to part a. above, determine the error ($\mathbf{e}$), the residual ($\mathbf{r}$).

(c.) Using MATLAB to invert this matrix, determine its condition number in the infinity norm.

(d.) Do your answers to part (b.) seem consistent with the condition number you found in part (c.)?

7. a. Show that for any matrices $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{C}^{m\times m}$, and any scalar α,

$$\kappa\left(\alpha\mathbf{A}\right) = \kappa\left(\mathbf{A}\right) \quad , \quad \kappa\left(\mathbf{A}^{-1}\right) = \kappa\left(\mathbf{A}\right) \quad \text{and} \quad \kappa\left(\mathbf{A}\,\mathbf{B}\right) \leq \kappa\left(\mathbf{A}\right)\kappa\left(\mathbf{B}\right)$$

b. Using the singular value decomposition, show that, for any $\mathbf{A} \in \mathbb{C}^{m\times n}$,

$$\kappa(\mathbf{Q}\,\mathbf{A}) = \kappa(\mathbf{A})$$

whenever the columns of $\mathbf{Q}$ are orthonormal, i.e. whenever $\mathbf{Q}^H\,\mathbf{Q} = \mathbf{I}$, and the condition number is computed in the Euclidean norm.

c. Why does the first of the results in part a. imply that $\det\mathbf{A}$ "close to" zero is a very *poor* test for nearly singular.

d. Why do the second and third of these results imply that, in the Euclidean norm, the condition of the normal equations formulation for solving the least squares problem, i.e.:

$$\mathbf{x} = \left(\mathbf{A}^H\,\mathbf{A}\right)^{-1}\mathbf{A}^H\,\mathbf{b}$$

may be as large as $\kappa(A)^3$, while the condition of solving the same problem by $\mathbf{QR}$ factorization, i.e.:

$$\mathbf{x} = \mathbf{R}^{-1}\left(\mathbf{Q}^H\,\mathbf{b}\right)$$

has condition exactly equal to $\kappa(\mathbf{A})$.

8. Consider the following "problem" from $\mathbb{R}^3$ to $\mathbb{R}^1$:

$$f(\mathbf{x}) = f(x,y,z) = (x+y) - z$$

together with the associated "algorithm"

$$\tilde{f}(\mathbf{x}) = \tilde{f}(x,y,z) = [fl(x) \oplus fl(y)] \ominus fl(z)$$

a. Simulate the results of this algorithm,for $x = 1$, $y = 0.005444\ldots$, and $z = 1$ in rounding, floating point machines using two through six decimal digits, and having an accumulator (ALU) of only the same length.

b. For each calclulation, determine

$$f(\mathbf{x}) \quad , \quad \tilde{f}(\mathbf{x}) \quad , \quad \text{and} \quad \frac{\|\,\tilde{f}(\mathbf{x}) - f(\mathbf{x})\,\|}{\|\,f(\mathbf{x})\,\|}$$

Discuss whether these results support or fail to support the statements:
(i.) $\tilde{f}$ is forward stable.
(ii.) $\tilde{f}$ is backward stable.

9. Consider the vectors

$$\mathbf{x} = [\ -1.19 \quad -2.20 \quad 0.986\ ]^T \quad \text{and} \quad \mathbf{y} = [\ -0.519 \quad 0.327 \quad 0.234\ ]^T$$

a. Simulate the calculation of $\mathbf{x}\,\mathbf{y}^T$ on a three-digit, decimal based computer which rounds all calculations, including intermediate ones.

b. Compare your solution in part a. to the exact (infinite precision) result.

c. Simulate the result of applying Gaussian elimination to this system on the same notional computer you used in part a. Based on your calculations, what would the theoretical rank be of the matrix?

d. What do your results in parts b. and c. say about whether this calculation is

(1.) Accurate (Forward Stable)

(2.) Backward Stable